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Reverse Plane Partitions and Tableau Hook Numbers

A. P. HILLMAN AND R. M. GRASSL

Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131

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The generating function of R. P. Stanley for reverse plane partitions on a tableau shape is obtained by a direct method that clearly shows the combinatorial significance of the hook numbers for the shape. The process generalizes the hooks into zigzag paths.

Generating functions for enumerating plane partitions generally are surprisingly simple in view of the manipulations that have been used to derive them. (For examples, see [1–3, 5, 8, 10–12].) This is also true of the theorem of Frame, Robinson, and Thrall [4, Theorem 1, p. 318] that the number of Young tableaux of a given shape with entries $1, 2,\ldots, n$ is $n!$ divided by the product of the hook numbers for the shape.

This paper obtains the generating function of Stanley for reverse plane partitions by a direct method that clearly shows the combinatorial significance of the hook numbers, while generalizing the hooks into zigzag paths. (Stanley's derivation is in [10]; also see [11, Proposition 18.3, p. 270].) As Stanley [12, (38), p. 82] has shown, this also provides a direct proof of the Frame–Robinson–Thrall result.

Tableaux were used by Young in his studies [13, 14] of irreducible representations of the symmetric groups of permutations. Nakayama introduced hook numbers [9, p. 166] in his extension of the work of Young.

1. ORDERING OF TABLEAU NODES

Let $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_t \geqslant 1$ with the p_i fixed integers. The *tableau shape* for $(p_1, p_2,\ldots, p_t)$ is the set S of nodes

$$\nu = (i, j) \qquad [i = 1, 2,\ldots, t \text{ and } j = 1, 2,\ldots, p_i].$$

As in a matrix, the i and j are the row and column numbers, respectively, of the node. The p_i are the lengths of the t rows; the length q_j of the jth column is the largest i with $p_i \geqslant j$.

The *hook* $H(\nu)$ for a node $\nu = (e, f)$ in S is the union

$$\{(e, j) \mid f \leqslant j \leqslant p_e\} \cup \{(i, f) \mid e < i \leqslant q_f\}.$$

The *hook number* $h(\nu)$ is the number $p_e - f + 1 + q_f - e$ of nodes in $H(\nu)$. The $s = p_1 + \cdots + p_t$ nodes in S are linearly ordered by saying that (i', j') *precedes* (i, j) if either $j < j'$ or $j = j'$ and $i > i'$. Then we label the nodes as $\nu_1, \nu_2, \ldots, \nu_s$ so that ν_k precedes ν_{k+1} for $1 \leqslant k < s$ and let $h_k = h(\nu_k)$.

We also introduce an ideal node ν_0, that is not in S, and say that ν_0 precedes each node of S.

2. Reverse Plane Partition

Let S be a tableau shape and $N = \{0, 1, 2, \ldots\}$. A *reverse plane partition* (rpp) of n on S is a mapping P from S to N such that n is the sum of all the entries $P(i, j)$ and P is a nondecreasing function of i for j fixed and of j for i fixed. (The word "reverse" indicates a change by Stanley to "nondecreasing" from the "nonincreasing" of earlier results.) The unique rpp of zero on S is called the *null* rpp.

Let α_n be the number of rpp of n on S and let $G(S)$ be the generating function $\sum_{n=0}^{\infty} \alpha_n x^n$. The following result is equivalent to Stanley's Proposition 18.3 [11, p. 270], in which the factor x^p occurs because he takes the p entries to be positive instead of nonnegative.

THEOREM 1. *$G(S) = \prod_{\nu \in S} [1 - x^{h(\nu)}]^{-1}$, where the $h(\nu)$ are the hook numbers for the nodes of S.*

In Sections 3–6 below, a generalization (Theorem 2 of Section 6) of Theorem 1 is proved. An example of the techniques involved is given in Section 7.

3. Hook Number Multiplicities for P

Clearly

$$\prod_{\nu \in S} [1 - x^{h(\nu)}]^{-1} = \prod_{k=1}^{s} [1 - x^{h_k}]^{-1} = \prod_{k=1}^{s} \sum_{m=0}^{\infty} x^{m h_k}$$

is the generating function for the number β_n of ordered s-tuples $M = (m_1, ..., m_s)$, of hook number *multiplicities* m_k, such that

$$\sum_{k=1}^{s} m_k h_k = n, \qquad m_k \in N = \{0, 1, ...\}. \tag{1}$$

We show that the α_n of Section 2 equals β_n by exhibiting below a one-to-one correspondence μ from the set A of all rpp of n on S onto the set B of all s-tuples M satisfying (1). Since $\alpha_0 = 1 = \beta_0$, we may restrict ourselves to nonnull rpp.

4. The Zigzag Path and the Derived rpp

Let P be an rpp of n on S with $n > 0$. Since P is nondecreasing in every column, $P(q_j, j) > 0$ for some j. Let c be the smallest such j and let $q_c = b$. Let $j_{b+1} = c$ and inductively assuming that $j_{b+1}, j_b, j_{b-1}, ..., j_{i+1}$ have been defined, let j_i be the smallest j with both $j \geqslant j_{i+1}$ and $P(i, j) = P(i-1, j)$, if such j exist. Ultimately, one reaches a row r such that j_{r+1} has been defined in this way and either $r = 1$ or there is no j with both $j \geqslant j_{r+1}$ and $P(r, j) = P(r-1, j)$; for this r let $j_r = p_r$. Now let $Z(P)$ be the zigzag path

$$Z(P) = \{(i, j) \mid b \geqslant i \geqslant r \text{ and } j_{i+1} \leqslant j \leqslant j_i\}. \tag{2}$$

We say that the ordered pair $\pi = (r, c)$ is the *pivot* of P. It is easily seen that π is a node of S, that the *path length* $z(P)$ of $Z(P)$ [i.e., the number of nodes in $Z(P)$] is the hook number $h(\pi)$, and that $P(\nu) > 0$ for $\nu \in Z(P)$.

The properties $P(i, j_i) = P(i-1, j_i)$ for $b \geqslant i > r$ and $P(i, j) > P(i-1, j)$ for any j with $j_{i+1} < j < j_i$ now imply that one obtains an rpp P' of $n - z(P)$ when all the entries of P along $Z(P)$ are depressed by 1 and the other entries are left unchanged; we call this P' the *derived* rpp for P.

To eliminate special cases, we say that the null rpp 0 has the ideal node ν_0 as its pivot, that $Z(0)$ is the null set, and that $z(0) = 0$.

We next show that the pivot $\pi' = (r', c')$ of the derived rpp P' either is π or precedes π. Since $P'(\nu) \leqslant P(\nu)$ for all ν in S, it is clear that $c' \geqslant c$. We assume that $c' = c$, since π' precedes π when $c' > c$.

For $b \geqslant i > r$, both (i, j_i) and $(i-1, j_i)$ are on the path $Z(P)$. Hence $P'(i, j_i) = P'(i-1, j_i)$ is implied by the same equality for P. These equalities and $j_{b+1} \leqslant j_b \leqslant \cdots \leqslant j_{r+1}$ guarantee that the path $Z(P')$ reaches at least to row r and so $r \geqslant r'$. [The j_i' for $Z(P')$ in the form (2) may be smaller than the j_i for $Z(P)$.] This shows that either $\pi' = \pi$ or π' precedes π.

5. Return Paths

Let $\pi = (r, c)$ be a node and let Q be an rpp whose pivot $\pi' = (r', c')$ either is π or precedes π. In this section, it is shown that an rpp P exists with π as pivot and with the derived rpp $P' = Q$.

The hypothesis on π and π' means that either (a) $c < c'$, or (b) $c = c'$ and $r \geqslant r'$.

We now seek the j_i of (2) for the desired P. Let $b = q_c$ and $j_r = p_r$. For $r < i \leqslant b$, we inductively assume that $j_r, j_{r+1}, j_{r+2}, \ldots, j_{i-1}$ have been defined and let j_i be the largest j with both $j \leqslant j_{i-1}$ and $Q(i, j) = Q(i-1, j)$. The existence of such j_i with $j_i \geqslant c$ is guaranteed in case (a) by the fact that $Q(i, c) = 0$ for $1 \leqslant i \leqslant b$ and in case (b) by the properties of the j_i' that characterize the path $Z(Q)$ in form (2) together with the relation $r \geqslant r'$. Finally, let $j_{b+1} = c$,

$$W = W(Q, \pi) = \{(i, j) \mid r \leqslant i \leqslant b \text{ and } j_i \geqslant j \geqslant j_{i+1}\},$$

and P be the function on S obtained by adding 1 to the entries of Q along W and leaving the other entries unaltered. The return path W has been defined so that P is an rpp. The equalities $P(i, j_i) = P(i-1, j_i)$ follow from the same equalities for Q. For any j with $j_i > j > j_{i+1}$ and $(i-1, j)$ in S, the inequality $P(i, j) > P(i-1, j)$ results from the addition of 1 to the entries of Q along W. Hence $W = Z(P)$. Then the pivot of P is π and $P' = Q$, as desired.

6. The Bijection μ

An rpp P of n on S determines a sequence $P, P', P'', \ldots, P^{(d)}$ where $P^{(k+1)}$ is the derived rpp for $P^{(k)}$ and $P^{(d)}$ is null. For $0 \leqslant k < d$, let z_k and π_k be the path length and pivot, respectively, of $P^{(k)}$. Then either $\pi_{k+1} = \pi_k$ or π_{k+1} precedes π_k. For $1 \leqslant i \leqslant s$, let m_i be the number of values of k in $\{0, 1, \ldots, d-1\}$ such that z_k is the hook number h_i for the node ν_i. Then the s-tuple $M = \mu(P) = (m_1, \ldots, m_s)$ of hook number multiplicities m_i satisfies $\sum_{i=1}^{s} m_i h_i = n$, since $P^{(k+1)}$ is an rpp of $n - z_0 - \cdots - z_k$ and $P^{(d)}$ is null.

It remains to show that μ is a bijection from A onto B, i.e., that any $M = (m_1, \ldots, m_s)$ with nonnegative m_i uniquely determines an rpp P with $\mu(P) = M$. Let M be given and $d = m_1 + \cdots + m_s$. Let $\pi_{d-1}, \pi_{d-2}, \ldots, \pi_0$ be the sequence of nodes in which the first m_1 terms equal ν_1, the next m_2 terms equal ν_2, etc. Since ν_i precedes ν_{i+1}, either $\pi_{k+1} = \pi_k$ or π_{k+1} precedes π_k. Now we can use the method of Section 5 to determine $P^{(d-1)}$ from the null rpp $P^{(d)}$ and π_{d-1}. Then $P^{(d-1)}$ and π_{d-2} determine $P^{(d-2)}$ and this continues until $P^{(0)} = P$ is obtained. Hence μ is a bijection.

We are now in a position to generalize on Stanley's result (Theorem 1 above). Let $\gamma(k, n)$ be the number of rpp P of n on S with the pivot of P in $\{\nu_0, \nu_1, \ldots, \nu_k\}$ and let the generating function for such rpp be $G_k = \sum_{n=0}^{\infty} \gamma(k, n)\, x^n$.

THEOREM 2. $G_0 = 1$ *and, for* $1 \leqslant k \leqslant s$,

$$G_k = \prod_{i=1}^{k} [1 - x^{h_i}]^{-1}. \tag{3}$$

Proof. Clearly the right side of (3) is the generating function for the s-tuples $M = (m_1, \ldots, m_s)$ with $m_i = 0$ for $i > k$. Since such s-tuples are the images under the bijection μ of the rpp P with pivots in $\{\nu_0, \ldots, \nu_k\}$, the theorem follows.

The case $k = s$ of Theorem 2 is Theorem 1.

COROLLARY. *For* $1 \leqslant k \leqslant s$, *the generating function for the* rpp P *of* n *on* S *with* ν_k *as pivot is*

$$G_k - G_{k-1} = x^{h_k} \prod_{i=1}^{k} [1 - x^{h_i}]^{-1}.$$

7. AN EXAMPLE

Next we illustrate the algorithms described above using the shape S with row lengths $(p_1, p_2, p_3) = (3, 3, 1)$, yielding $s = p_1 + p_2 + p_3 = 7$ nodes, and the multiplicity 7-tuple $(m_1, \ldots, m_7) = (0, 1, 2, 1, 2, 1, 1)$.

Each of the following arrays I, L, and M has seven entries in the positions of the seven nodes of S; the subscripts i of ν_i (indicating the ordering of the nodes) are given in I, the hook lengths h_i are in L, and their multiplicities m_i are in M:

$$\begin{array}{ccc} I & L & M \\ \begin{matrix} 5 & 3 & 1 \\ 6 & 4 & 2 \\ 7 & & \end{matrix} & \begin{matrix} 5 & 3 & 2 \\ 4 & 2 & 1 \\ 1 & & \end{matrix} & \begin{matrix} 2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & & \end{matrix} \end{array}$$

The 7-tuple M shows that $d = m_1 + \cdots + m_7 = 8$ and then that the pivots $\pi_7, \ldots, \pi_0$ are $\nu_2, \nu_3, \nu_3, \nu_4, \nu_5, \nu_5, \nu_6, \nu_7$. The null rpp $P^{(8)}$ and

π_7 determine $P^{(7)}$, then $P^{(7)}$ and π_6 determine $P^{(6)}$, and so on, with results as follows.

$$\begin{array}{cccccccc}
P^{(7)} & P^{(6)} & P^{(5)} & P^{(4)} & P''' & P'' & P' & P \\
0\,0\,0 & 0\,1\,1 & 0\,1\,2 & 0\,1\,2 & 1\,2\,3 & 1\,2\,4 & 1\,2\,4 & 1\,2\,4 \\
0\,0\,1 & 0\,1\,1 & 0\,2\,2 & 0\,3\,3 & 1\,3\,3 & 2\,4\,4 & 3\,5\,5 & 3\,5\,5 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4
\end{array}$$

The path $Z(P^{(k)})$ consists of the ν with $P^{(k+1)}(\nu) = P^{(k)}(\nu) - 1$. Also, $n = m_1h_1 + \cdots + m_7h_7 = 24 = P(\nu_1) + \cdots + P(\nu_7)$.

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